JOURNAL OF COMPUTATIONAL PHYSICS 15, 286–292 (1974)

Numerical Solution of a Singular Integral Equation in Random Rough Surface Scattering Theory

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Received January 21, 1974

A one-dimensional singular integral equation which appeared in a previous paper on random rough surface scattering theory (*J. Math. Phys.* **13**, 1903 (1972)) is solved numerically using quadratic splines. Its solution yields an approximation to the coherent (specular) scattered intensity for plane wave incidence on the surface. This approximate scattered intensity is plotted versus the Rayleigh factor $\Sigma = k_0 \sigma \cos \theta_i$, where k_0 is the wavenumber of the incident plane wave, σ is the surface root mean square height, and θ_i is the angle of the incident plane wave. For values of $\Sigma > 1$ this approximation yields more coherent intensity than the Kirchhoff approximation.

1. INTRODUCTION

Some recent papers [1–3] have considered scattering from a Gaussian distributed random rough surface. Feynman diagramlike techniques were used to simplify the resulting integral equations for the moments of the surface Green's function, and to systematize the approximations to these equations. In particular, an integral equation, called the Dyson equation, was derived for the first moment (mean) of a function linearly related to the surface Green's function. Using the planar translational invariance of the problem, this Dyson equation was written as a one-dimensional singular integral equation. Its solution is complicated by the fact that the Born (or inhomogeneous) term and the kernel of the equation are (related) infinite series of terms (connected diagrams) which are not known in closed form. In this paper we study numerically the lowest order approximate solution of this Dyson equation obtained by approximating the Born term and the kernel by the first term in their series expansion. This yields an integral equation of the following form [4] for the scattering amplitude τ^+

$$\tau^+(k_z, k_z') = C(k_z - k_z') + \frac{K^2}{\pi i} \int_{-\infty}^{\infty} \frac{C(k_z - k_z'')}{(k_z'')^2 - K^2 - i\epsilon} P\left(\frac{1}{k_z''}\right) \tau^+(k_z'', k_z')\, dk_z'', \quad (1)$$

where k_z, k_z', etc., are the z-components of wavenumbers, $K^2 = k_0^2 - k_\perp^2$, with

k_0 the wavenumber of the plane wave incident on the surface and $k_\perp$ the transverse component of the wavenumber vector, i.e., $k_\perp = (k_x, k_y)$. Here the dependence of the function τ^+ on k_0 and $k_\perp$ is suppressed. P indicates the Cauchy principle value distribution, and the limit $\epsilon \to 0^+$ is understood. The characteristic function $C(k_z)$ of the Gaussian surface distribution is given by

$$C(k_z) = \exp[-(1/2)\, \sigma^2 k_z^2], \tag{2}$$

where σ is the root mean square height of the surface. The function $C(k_z)$ is the first term in the infinite series expansion of the Born term and the kernel of the exact Dyson equation, and the approximation $\tau^+(k_z, k_z') \approx C(k_z - k_z')$ yields the Kirchhoff approximation [5]. We want to calculate the specular (coherent) intensity I given by

$$I(k_\perp, k_{i\perp}) = |\, \tau^+(K, -K)|^2\, \delta(k_\perp - k_{i\perp}) \tag{3}$$

for a wave scattered in the $k_\perp$ direction due to an incident wave in the $k_{i\perp}$ direction. Here $\delta(k_\perp)$ stands for the two-dimensional (transverse) Dirac delta function, and the functional value $\tau^+(K, -K)$ is found by putting the solution of (1) "on-shell," i.e., by setting $k_z = K$ and $k_z' = -K$. This brief introduction is intended only to summarize the background. A complete discussion can be found in Ref. [1].

In Section 2 we discuss the numerical solution of (1) using scaling and a quadratic spline approximation. The numerical results are presented in Section 3, where $|\, \tau^+\, |^2$ is plotted versus the parameter $\Sigma = k_0\sigma \cos\theta_i$, θ_i being the incidence angle of the plane wave. Although we confine our discussion to the scalar case for a hard (Neumann) boundary [1], similar matrix sets of equations arise in elastic media with a stress-free random boundary [2], and in electromagnetic media with a perfectly conducting random boundary [3].

2. Numerical Solution

In this section we discuss the numerical solution of (1). First, scale the equation. That is, consider K to be a fixed parameter of the problem, and scale the other parameters in the equation with respect to K. Define ξ by

$$k_z = k_0 \cos\theta = k_0 \xi \cos\theta_i = K\xi, \tag{4}$$

with the on shell value $\xi = 1$. Substituting (4) in (1) and setting

$$T(\xi, \xi') = \tau^+(K\xi, K\xi'), \tag{5}$$

$$C_0(\xi) = C(K\xi), \tag{6}$$

we write (1) as

$$T(\xi, \xi') = C_0(\xi - \xi') + (\pi i)^{-1} \int_{-\infty}^{\infty} \frac{C_0(\xi - \xi'')}{\xi''^2 - 1 - i\epsilon} P\left(\frac{1}{\xi''}\right) T(\xi'', \xi')\, d\xi'', \tag{7}$$

and, from (3), we wish to calculate $|T(1, -1)|^2$. Rewrite (7) so that the kernel will have three principal value singularities by using the operator relation [6, 7]

$$(\xi''^2 - 1 - i\epsilon)^{-1} = P(\xi''^2 - 1)^{-1} + (\pi i/2)\{\delta(\xi'' - 1) + \delta(\xi'' + 1)\}, \tag{8}$$

so that (7) becomes

$$T(\xi, \xi') = C_0(\xi - \xi') + (1/2)\{C_0(\xi - 1)\, T(1, \xi') - C_0(\xi + 1)\, T(-1, \xi')\} + \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{C_0(\xi - \xi'')\, T(\xi'', \xi')\, d\xi''}{\xi''(\xi''^2 - 1)}. \tag{9}$$

Letting successively $\xi = +1$ and $\xi = -1$ in (9), algebraically "solving" the resulting set of equations for $T(1, \xi')$ and $T(-1, \xi')$ in terms of still unknown integrals and resubstituting the results into (9) yields the result

$$T(\xi, \xi') = B(\xi, \xi') + \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{B(\xi, \xi'')\, T(\xi'', \xi')\, d\xi''}{\xi''(\xi''^2 - 1)}, \tag{10}$$

where the Born term B is defined as

$$\begin{aligned} B(\xi, \xi') = C_0(\xi - \xi') + \Delta\{&C_0(\xi - 1)[(3/2)\, C_0(1 - \xi') - (1/2)\, C_0(2)\, C_0(1 + \xi')] \\ &- C_0(\xi + 1)[(1/2)\, C_0(1 + \xi') + (1/2)\, C_0(2)\, C_0(1 - \xi')]\}, \end{aligned} \tag{11}$$

with

$$\Delta = 2[3 + C_0^2(2)]^{-1}. \tag{12}$$

Hence, we want the solution of (10) in order to calculate $|T(1, -1)|^2$.

To solve (10) numerically, we make use of the following observation:

Let $m \geqslant 2$ be an even integer, $h = 1/m$, L an integer >1, $N = mL$, $\xi_k = kh$ for $k = 0, \pm 1, \pm 2, \ldots, \pm N$ so that $L = \xi_N$, $-L = \xi_{-N}$, $0 = \xi_0$, $1 = \xi_m$, and $-1 = \xi_{-m}$. Let $\varphi(y)$ be a real function defined on $[-L, L]$, and define for $-L \leqslant y \leqslant L$

$$S_h(y) = \sum_{k=-N}^{N} \varphi(\xi_k)\, s_k(y),$$

where each quadratic spline $s_k(y)$ is defined in $[-L, L]$ as follows [8]:

(a) *If k is odd, then*

$$s_k(y) = \begin{cases} (-h^2)^{-1}\,(y - \xi_{k-1})(y - \xi_{k+1}) & \textit{for} \quad y \in [\xi_{k-1}, \xi_{k+1}], \\ 0 & \textit{elsewhere in } [-L, L]. \end{cases}$$

(b) *If k is even, $-N < k < N$, then*

$$s_k(y) = \begin{cases} (2h^2)^{-1}\,(y - \xi_{k-2})(y - \xi_{k-1}) & \textit{for} \quad y \in [\xi_{k-2}, \xi), \\ (2h^2)^{-1}\,(y - \xi_{k+1})(y - \xi_{k+2}) & \textit{for} \quad y \in [\xi_k, \xi_{k+2}], \\ 0 & \textit{elsewhere in } [-L, L]. \end{cases}$$

(c) $$s_{-N}(y) = \begin{cases} (2h^2)^{-1}\,(y - \xi_{-N+1})(H \wedge y - \xi_{-N+2}) & \textit{for} \quad y \in [\xi_{-N}, \xi_{-N+2}], \\ 0 & \textit{elsewhere in } [-L, L]. \end{cases}$$

(d) $$s_N(y) = \begin{cases} (2h^2)^{-1}\,(y - \xi_{N-2})(y - \xi_{N-1}) & \textit{for} \quad y \in [\xi_{N-2}, \xi_N], \\ 0 & \textit{elsewhere in } [-L, L]. \end{cases}$$

Then if p is even, $-N < p \leqslant N$, on the interval $[\xi_{p-2}, \xi_p]$, $S_h(y)$ coincides with a second degree polynomial interpolating $\varphi(y)$ at the points ξ_{p-2}, ξ_{p-1}, and ξ_p.

To verify this observation, note that on the interval $[\xi_{p-2}, \xi_p]$, where $-N < p \leqslant N$, we have

$$\begin{aligned} S_h(y) &= \varphi(\xi_{p-2})\, s_{p-2}(y) + \varphi(\xi_{p-1})\, s_{p-1}(y) + \varphi(\xi_p)\, s_p(y) \\ &= (2h^2)^{-1}\,(y - \xi_{p-1})(y - \xi_p)\,\varphi(\xi_{p-2}) \\ &\quad - (h^2)^{-1}\,(y - \xi_{p-2})(y - \xi_p)\,\varphi(\xi_{p-1}) \\ &\quad + (2h^2)^{-1}\,(y - \xi_{p-2})(y - \xi_{p-1})\,\varphi(\xi_p), \end{aligned}$$

so that

$$S_h(\xi_{p-2}) = \varphi(\xi_{p-2}), \qquad S_h(\xi_{p-1}) = \varphi(\xi_{p-1}), \qquad \text{and} \qquad S_h(\xi_p) = \varphi(\xi_p).$$

Now, given the integral equation (10), with the unknown function T and parameter ξ', choose m and L as above and replace (10) by

$$T(\xi, \xi') = B(\xi, \xi') + \frac{1}{\pi i}\, P \int_{-L}^{L} \frac{B(\xi, \xi'')\, T(\xi'', \xi')\, d\xi''}{\xi''(\xi''^2 - 1)}, \tag{13}$$

with singularities at 0, 1, and -1. Let h, N, and $\xi_k (k = 0, \pm 1, \ldots, \pm N)$ be as above. For each ξ, ξ' we regard $B(\xi, \xi'')\, T(\xi'', \xi')$ in (13) as the $\varphi(y)$ of the above observation, and we replace it by $S_h(y)$. Thus, (13) is replaced by

$$T(\xi, \xi') = B(\xi, \xi') + \frac{1}{\pi i} \sum_{k=-N}^{N} B(\xi, \xi_k)\, T(\xi_k, \xi')\, a_k, \tag{14}$$

with

$$a_k = P\int_{-L}^{L} \frac{s_k(y)}{y(y^2-1)}\,dy, \qquad k = 0, \pm 1, \pm 2,..., \pm N. \tag{15}$$

Next, set the parameter $\xi' = -1$ since we only need $T(\xi, -1)$. To solve (14) numerically, set there $\xi = \xi_j$ ($j = 0, \pm 1, \pm 2,..., \pm N$), and obtain the system of $2N + 1$ linear equations in the $2N + 1$ unknowns $T(\xi_j, -1)$:

$$\sum_{k=-N}^{N} b_{jk} T(\xi_k, -1) = -B(\xi_j, -1), \tag{16}$$

where, for $j, k = 0, \pm 1, \pm 2,..., \pm N$,

$$b_{jk} = (\pi i)^{-1} a_k B(\xi_j, \xi_k) - \delta_{jk}, \tag{17}$$

δ_{jk} being the Kronecker delta

$$\delta_{jk} = \begin{cases} 1 & \text{for} \quad j = k, \\ 0 & \text{for} \quad j \neq k. \end{cases}$$

We can derive from our definition of the a_k

$$a_0 = 0,$$
$$a_{-k} = -a_k, \qquad k = 1, 2,..., N.$$

Performing the integrations (15) (assuming $m > 4$) we obtain

(a) If k is odd, $k > 0$

$$\begin{aligned} a_k = &-(2h^2)^{-1}[1-(k+1)h][1-(k-1)h]\ln\{[1-(k+1)h]/[1-(k-1)h]\} \\ &-(2h^2)^{-1}[1+(k+1)h][1+(k-1)h]\ln\{[1+(k+1)h]/[1+(k-1)h]\} \\ &+(k^2-1)\ln\{(k+1)/(k-1)\}. \end{aligned}$$

(b) If k is even, $2 \leqslant k < m-2$ or $m+2 < k \leqslant N-2$,

$$\begin{aligned} a_k = &(3/2h)\{(1-kh)\ln(1-kh)-(1+kh)\ln(1+kh)\} \\ &+3k\ln(kh)+(1/2)(k-2)(k-1)\ln[(k-2)h] \\ &-(1/2)(k+2)(k+1)\ln[(k+2)h] \\ &+(4h^2)^{-1}\{[1+(k+1)h][1+(k+2)h]\ln[1+(k+2)h] \\ &+[1-(k+1)h][1-(k+2)h]\ln[1-(k+2)h] \\ &-[1+(k-1)h][1+(k-2)h]\ln[1+(k-2)h] \\ &-[1-(k-1)h][1-(k-2)h]\ln[1-(k-2)h]\}, \end{aligned}$$

$$a_m = (2h^2)^{-1}\{(2+h)(1+h)\ln(1+h) - (2-h)(1-h)\ln(1-h) + (1-h)(1-2h)\ln(1-2h) - (1+h)(1+2h)\ln(1+2h)\},$$

$$a_{m\pm 2} = \pm 3\ln 2 + (2h^2)^{-1}\{\mp(1\pm 3h)(1\pm 4h)\ln(1\pm 4h) - 6h(1\pm h)\ln(1\pm h) \pm (2\pm 13h + 18h^2)\ln(1\pm 2h)\},$$

$$a_N = +(4h^2)^{-1}[(N+2)h+1][(N+1)h+1]\ln\{[Nh+1]/[(N+2)h+1]\} + (4h^2)^{-1}[(N+2)h-1][(N+1)h-1]\ln\{[Nh-1]/[(N+2)h-1]\} - (2h^2)^{-1}(N+2)(N+1)h^2\ln\{N/(N+2)\}.$$

Using these values for the a_k, (16) was inverted taking $m = 6$ and $N = 30$. The results are discussed in the next section.

3. Numerical Results

Figure 1 is a plot of the natural logarithms of $|T(1,-1)|^2$, $B^2(1,-1)$ and $C_0^2(2)$ versus the parameter $\Sigma = k_0\sigma\cos\theta_i$ for Σ varying over $[0, 2]$. Σ is called the Rayleigh roughness parameter [5]. The function $C_0^2(2) = \exp(-\Sigma^2)$ is the Kirchhoff approximation to the specular (coherent) intensity [5]. The function $B(1,-1)$ was derived by removing the singularities in our original integral equation

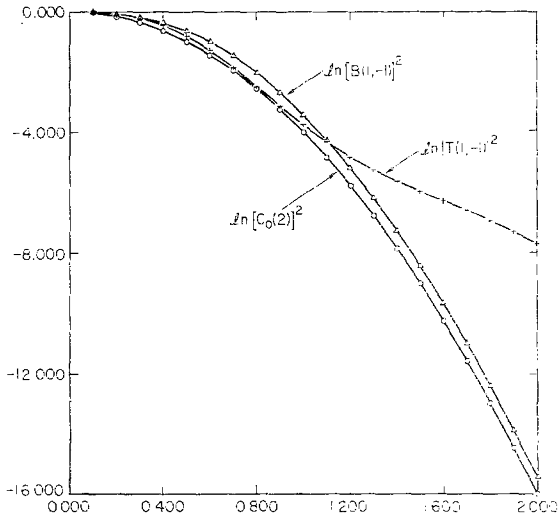

Fig. 1. Plot of the natural logarithms of $|T^+(1,-1)|^2$, $B^2(1,-1)$ and $C_0^2(2)$ versus $\Sigma = k_0\sigma\cos\theta_i$.

and is defined by (11). $T(1, -1)$ is an approximate solution of (10) and was obtained by solving (16). Its relation to our original amplitude τ^{+} is given by (5).

The results indicate that the removal of the singularities to generate a new inhomogeneous term of the integral equation (i.e., going from C_0 in (7) to B in (10)) has no great effect, at least on shell, since $C_0(2)$ and $B(1, -1)$ are nearly equal over the full range of Σ. Indeed, up to about $\Sigma = 1$, solving the integral equation makes no appreciable difference in the resulting coherent intensity. For larger values of Σ, however, there is considerable difference in T and B. Thus, the Kirchhoff approximation for the coherent (specular) intensity yields virtually the same results as our lowest order approximate solution of the Dyson equation, $T(1, -1)$, up to $\Sigma = 1$. For $\Sigma > 1$, $T(1, -1)$ yields more coherent (specular) intensity than the Kirchhoff approximation.

Acknowledgments

We are grateful to Professor K. Atkinson for his kind help, to Miss Mary Ellen Mosko of the Naval Research Laboratory Research Computation Center for programming the original equations, and to Mr. Charles Smith and Mrs. Luise Schuetz for additional numerical assistance.

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